Math 429 - Exercise Sheet 7

1. Show that if $W \subseteq V$ are representations of a Lie algebra $\mathfrak{g}$, then the associated s.i.b.f.'s satisfy

$$(x, y)_V = (x, y)_W + (x, y)_{V/W}$$

Solution. Let $(w_1, \dots, w_k)$ be a basis of W and complete it to a basis $\mathcal{B} = (w_1, \dots, w_k, v_{k+1}, \dots, v_n)$ of V. The images of $v_1, \dots, v_n$ in V/W form a basis $\overline{\mathcal{B}}$ of this quotient. Then, for any $x \in \mathfrak{g}$ we write the associated operator $\phi_x \in \mathfrak{gl}(V)$ with respect to the basis $\mathcal{B}$ and we get

$$\phi_x = \begin{bmatrix} \phi_x\big|_W & * \\ 0 & \overline{\phi_x} \end{bmatrix},$$

where $\overline{\phi_x}$ is the induced linear operator on V/W, written with respect to the basis $\overline{\mathcal{B}}$. Then the claim follows.

2. Show that if $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{i}^\perp$ (with respect to any s.i.b.f.) is also an ideal.

Solution. Let $x \in \mathfrak{i}^\perp$ and $y \in \mathfrak{g}$ and $z \in \mathfrak{i}$. Then

$$([x, y], z) = -([x, z], y) = ([z, x], y) = -([z, y], x) = 0,$$

where in the last equality we used the fact thet $\mathfrak{i}$ is an ideal of $\mathfrak{g}$. Then $[x, z] \in \mathfrak{i}^\perp$.

3. Show that if $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then

$$(x, y)_\mathfrak{i} = (x, y)_\mathfrak{g}$$

for all $x, y \in \mathfrak{i} \subseteq \mathfrak{g}$.

Solution. For any $x \in \mathfrak{i}$, write as $\mathrm{ad}_{x,\mathfrak{g}}$ and $\mathrm{ad}_{x,\mathfrak{i}}$ the adjoint operator in $\mathfrak{g}$ and $\mathfrak{i}$ respectively. Then, choosing a basis appropriately, we have

$$\mathrm{ad}_{x,\mathfrak{g}} \begin{bmatrix} \mathrm{ad}_{x,\mathfrak{i}} & * \\ 0 & 0 \end{bmatrix},$$

and the claim follows.

4. Prove that $\mathfrak{sl}_n$ is a simple Lie algebra (*Hint: take any non-zero $X \in \mathfrak{sl}_n$, and show that you can obtain any E_{ij}, $i \neq j$ from X by suitably taking commutators*).

Solution. Let $\mathfrak{i} \subset \mathfrak{sl}_n$ be a nonzero ideal and let $X \in \mathfrak{i}$ be a nonzero element. Denoting with $E_{i,j}$ the elementary matrices, we prove that any element of the basis

$$(E_{i,j})_{i \neq j, 1 \leq i,j \leq n} \cup (E_{i,i} - E_{nn})_{1 \leq i \leq n-1} \tag{1}$$

of $\mathfrak{sl}_n$ can be reached from X by taking a finite number of commutators. This implies that $\mathfrak{i} = \mathfrak{sl}_n$. First, fix $i \neq j$ and we prove that $E_{i,j} \in \mathfrak{i}$. We have

$$[X, E_{i,j}] = \begin{bmatrix} 0 & \dots & 0 & \overbrace{x_{1,i}}^{j^{\text{th}} \text{ coloumn}} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & x_{i-1,i} & 0 & \dots & 0 \\ \overbrace{-x_{j,1}}^{i^{\text{th}} \text{ row}} & \dots & -x_{j,j-1} & x_{i,i} - x_{j,j} & -x_{j,j+1} & \dots & -x_{j,n} \\ 0 & \dots & 0 & x_{i+1,i} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & x_{n,i} & 0 & \dots & 0 \end{bmatrix}. \tag{2}$$

If $x_{i,j} = 0$ for every $i \neq j$ (that is, if $X \in \mathrm{Span}(E_{i,i} - E_{nn})_{1 \le i \le n-1}$), then we can suppose that $x_{i,i} - x_{j,j} \neq 0$ and (2) gives $[X, E_{i,j}] = (x_{i,i} - x_{j,j})E_{i,j}$.
Viceversa, assume that $x_{j,i} \neq 0$ for some $i \neq j$. Then we can take another bracket as in (2) and get $[[X, E_{i,j}], E_{i,j}] = -2x_{j,i}E_{j,i}$. This proves that $E_{i,j} \in \mathfrak{i}$. Finally, the other elements in the basis (1) can be obtained by

$$[E_{i,j}, E_{j,k}] = \begin{cases} E_{i,k} & i \neq k \\ E_{i,i} - E_{j,j} & i = k, \end{cases}$$

which completes the proof.

5. Because of the previous problem, Lemma 2 implies that the Killing form of $\mathfrak{sl}_n$ must be equal to a constant times the s.i.b.f. $(X, Y) \mapsto \mathrm{tr}(XY)$. Calculate the constant in question.

Solution. It is enough to check both s.i.b.f.'s on the matrix $H = \mathrm{Diag}(1, -1, 0, \dots, 0) \in \mathfrak{sl}_n$. We have $\mathrm{tr}(H^2) = 2$. On the other side, the operator ad_H is diagonal with respect to the basis (1). In fact,

$$\mathrm{ad}_H(E_{i,i} - E_{n,n}) = 0 \text{ for all } i = 1, \dots, n-1,$$

and

$$\mathrm{ad}_H(E_{i,j}) = (\epsilon_i - \epsilon_j)E_{i,j} \text{ for all } 1 \le i, j \le n, i \neq j,$$

where

$$\epsilon_k = \begin{cases} 1 & k = 1 \\ -1 & k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then (after reordering the basis) $\mathrm{ad}_H = \mathrm{Diag}(\overbrace{1, \dots, 1}^{2(n-2)}, \overbrace{-1, \dots, -1}^{2(n-2)}, 2, -2, 0, \dots, 0)$, and $\mathrm{tr}(\mathrm{ad}_{\mathrm{H}}^2) = 4n$. Thus,

$$\mathrm{tr}(\mathrm{ad}_{\mathrm{X}} \, \mathrm{ad}_{\mathrm{Y}}) = 2n \, \mathrm{tr}(XY)$$

for all $X, Y \in \mathfrak{sl}_n$.

6. For $\mathfrak{g} \in \{\mathfrak{o}_n, \mathfrak{sp}_{2n}\}$, check that the s.i.b.f. $(X, Y) \mapsto \mathrm{tr}(XY)$ is non-degenerate. Here we are considering the trace on $n \times n$ matrices (in the case $\mathfrak{o}_n$) and on $2n \times 2n$ matrices (in the case $\mathfrak{sp}_{2n}$).

Solution. Suppose that $X = (x_{i,j}) \in \mathfrak{o}_n$ is such that $\mathrm{tr}(XY) = 0$ for all $Y \in \mathfrak{o}_n$. In particular,

$$0 = \mathrm{tr}(X(E_{i,j} - E_{j,i})) = 2x_{j,i}.$$

for all $j \neq i$, so that $X = 0$.

Now consider $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sp}_{2n}$, where $D = -A^t, B = B^t, C = C^t$ and suppose that $\mathrm{tr}(XY) = 0$ for all $Y \in \mathfrak{sp}_{2n}$. In particular,

$$0 = \mathrm{tr}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & -(A')^t \end{bmatrix}\right) = 2\,\mathrm{tr}(AA')$$

for all $A' \in \mathfrak{gl}_n$. This implies that $A = 0$, as in the Lecture notes. Similarly,

$$0 = \mathrm{tr}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ C' & 0 \end{bmatrix}\right) = 2\,\mathrm{tr}(BC')$$

for every symmetric matrix C'. Choosing $C' = E_{i,i} + E_{n-i,n-i}$ shows that $B = 0$. An analogous procedure yields $C = 0$.

7. If X is an upper triangular $n \times n$ matrix, prove that

$$\mathrm{ad}_X : \mathfrak{gl}_n \to \mathfrak{gl}_n, \qquad \mathrm{ad}_X(Y) = [X, Y]$$

is a nilpotent operator (thus establishing the first blue claim in Lecture 7).

Solution. Let $X \in \mathfrak{gl}_n$ be such that $X^N = 0$ and let Y be any matrix in $\mathfrak{gl}_n$. The iterated bracket

$$\overbrace{[X, [X, \dots, [X}^{M}, Y] \dots]] \tag{3}$$

is a sum of monomials $X^a Y X^b$ such that $a + b = M - 1$. If m denotes the smallest of these exponents $\{a, b\}$, it follows that $m \geq \frac{M}{2} - 1$. Then, for $M >> 0$ we have $m > N$, and the bracket (3) vanishes.

(*) Prove the first blue claim in Subsection 7.3 of the lecture notes: "*However, you can prove by analogy with Theorem 11 that any $y \in \mathfrak{g} \backslash rad(\mathfrak{g})$ also sends W to W*".